International Journal of Multidisciplinary Research and Modern Education (IJMRME)
Impact Factor: 7.315, ISSN (Online): 2454 - 6119
(www.rdmodernresearch.org) Volume 7, Issue 1, 2021

APPROACHES ON FUZZY DIFFERENTIAL EQUATIONS

M. Vasuki* & A. Dinesh Kumar**

* Assistant Professor, Department of Mathematics, Srinivasan College of Arts & Science, Perambalur, Tamilnadu

** Associate Professor, School of Engineering, Pokhara University and Madan Bhandari Memorial Academy Nepal (Joint Constitute), Morang, Nepal

Cite This Article: M. Vasuki & A. Dinesh Kumar, "Approaches on Fuzzy Differential Equations", International Journal of Multidisciplinary Research and Modern Education, Volume 7, Issue 1, Page Number 37-55, 2021.

Abstract:

This paper may be used as reference for graduate students interested in fuzzy differential equations and researchers working in fuzzy sets and systems, dynamical systems, uncertainty analysis, and applications of uncertain dynamical systems. Beginning with a historical overview and introduction to fundamental notions of fuzzy sets, including different possibilities of fuzzy differentiation and metric spaces, this paper moves on to an overview of fuzzy calculus thorough exposition and comparison of different approaches. Innovative theories of fuzzy calculus and fuzzy differential equations using fuzzy bunches of functions are introduced and explored.

Introduction:

One of the branches of Mathematics conveyed clearly in the Principal language of Science called "Differential Equations" plays an important role in Science, Engineering and Social Sciences. Let us analyse a few of the examples cited below. Suppose that there are two living species which depend for their survival on a common source of food supply. This fact results in a competition in consuming the available food. The phenomenon is commonly noticed in the plant life having common supply of water, fertilizer and minerals. However whenever the competition between two species begins. The growth rate of one is retarded and we can note that the rate of retardation is naturally proportional to the size of the other Species present at time t. This situation can be expressed as a Mathematical model whose solution would help us to determine the time at which one species would become extinct. Several diseases are caused by spread of an infection. Suppose that the susceptible population of a town is p. One person gets the infection. Because of contact another susceptible person is also infected. This process continues to cover the entire susceptible population with some assumptions to simplify the mathematical considerations. This situation can be framed into a mathematical model and a solution can be determined which would provide information regarding the spread of the epidemic in the town.

If a dead body is brought for a medical examination at a particular time, the exact time of death can be determined by noting the temperature of the body at various time intervals, formulating it into a mathematical problem with available initial conditions and then solving it. From the above examples it is found that the mathematical formulation to all situations turns out to be differential equations. Thus the latent significance of differential equations in studying physical phenomena becomes apparent. This branch of Mathematics called Differential Equations is like a bridge linking Mathematics and Science with its applications. Hence it is rightly considered as the language of Sciences. Galileo once conjectured that the velocity of a body falling from rest is proportional of the distance fallen. Later he decided that is proportional to the time instead. Each of these statements can be formulated as an equation involving the rate of change of an unknown function and is therefore an example of what Mathematician call a Differential Equations. Thus $\frac{ds}{dt} = kt$ is a differential equation which gives velocity of a falling body from a distance 's' proportional to the time 't'.

An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called Differential Equations. In modeling of real physical phenomena, differential equations play an important role in many areas of discipline, namely in economics, science and engineering. Many experts in such areas extensively use differential equations in order to make some problems under study more understandable. In many cases, information about the physical phenomena involved is always pervaded with uncertainty. According to Diniz et al. (2001), the uncertainty can arise in the experiment part, data collection, measurement process as well as when determining the initial values. These are patently obvious when dealing with "living" materials, such as soil, water and microbial populations (Ahamed and De Baets 2009). Classical mathematics, however, cannot cope with this situation. Therefore, it is necessary to have some mathematical apparatus in order to understand this uncertainty. Various theories exist for describing this uncertainty and the most popular one is fuzzy set theory (Zadeh 1965).

Today, the study of differential equations with uncertainty is rapidly growing as a new area in fuzzy analysis. The terms such as "fuzzy differential equation", "fuzzy differential inclusion", and "set differential equation" are used interchangeably in referring to differential equations with fuzzy initial values or fuzzy boundary values or even differential equations dealing with functions on the space of fuzzy numbers (Buckley and Feuring 2000: Hullermeier 1997; Kaleva 1987; Seikkala 1987; Laksmikantham 2004).

Preliminaries:

Definition: A fuzzy set is a generalisation of a classical set that allows membership function to take any value in the unit interval [0, 1].

Definition: Let U be a universal set. A fuzzy set A in U is defined by a membership function A(x) that maps every element in U to the unit interval [0, 1]. A fuzzy set A in U may also be presented as a set of ordered pairs of a generic element x and its membership value, as shown in the following equations.

$$A = \{(x, A(x) | x \in u\}$$

Definition: Let A be a fuzzy set defined in U. The support of A is the crisp set of all elements in U such that the membership function of A is non- zero, that is,

$$\text{Supp}\,(A) = \{x \in U | A(x) > 0\}$$

Definition: Let A be a fuzzy set defined in U. The core of A is the crisp set of all elements in U such that the membership value of A is 1, which is

$$\text{Core}\,(A) = \{x \in U \mid A(x) = 1\}$$

Definition: Let A be a fuzzy set defined in R. A is called a fuzzy interval if.

- A is normal, that is there exists $x_0 \in R$ such that $A(x_0) = 1$;
- A is convex, that is for all $x, y \in R$ and $0 \leq n \leq 1$, it holds that ,
$$A(\lambda x + (1-\lambda)y) \geq \min\,(A(x), A(y));$$
- A is upper semi-continuous, that is for any $x_0 \in R$, it holds that
$$A(x_0) \geq \lim_{x \to x_0^{\pm}} A(x);$$
- $[A]^0 = \overline{\{x \varepsilon R \mid A(x) \geq \alpha\}}$ is a compact subset of R.

Definition: Let A be a fuzzy interval defined in R. The α-cut of A is the crisp set $[A]^0$ that contains all elements in R such that the membership values of A is greater than or equal to α that is

$$[A]^u = \{x \in R | A(x) \geq \alpha\},\ \alpha \in (0,1]$$

For a fuzzy interval A. its α- cuts are closed intervals in R and we denote them by

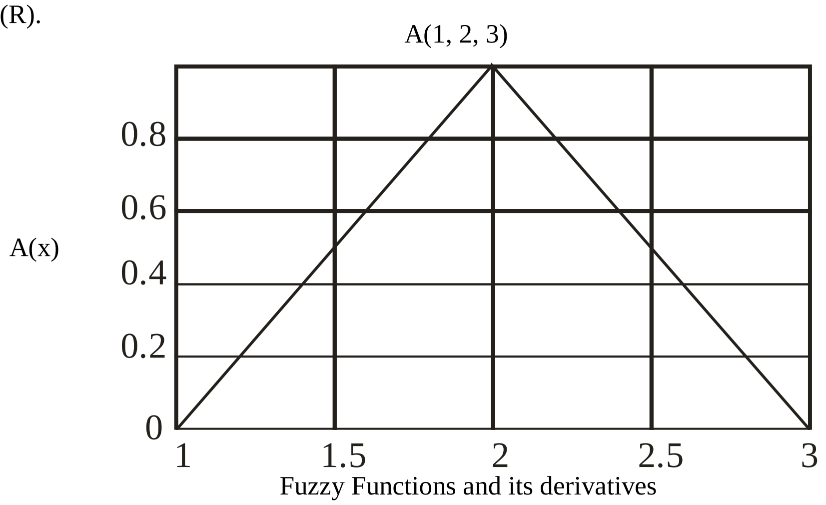
$$[A]^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}],\ \alpha \in (0,1]$$

Definition: A fuzzy interval A is called a triangular fuzzy interval if its membership function has the following form

$$A(x) \quad = \quad \begin{cases} 0, & \text{if } x < a \\ \dfrac{x - a}{b - a}, & \text{if } a \leq x \leq b \\ \dfrac{c - x}{c - b}, & \text{if } b \leq x \leq c \\ 0, & \text{if } x > c \end{cases}$$

and it's α-cuts are simply.

$[A]^{\infty} = [a + \infty(b - a), c - \infty(c - b)]$, $\infty \in (0, 1]$. This definition asserts that the triangular fuzzy interval A is defined by three numbers a < b < c, where the core of A is b and its support is the interval (a, c). Figure 1 shows the example of triangular fuzzy interval. In this paper the set of all triangular fuzzy intervals will be denoted by f(R).



Fuzzy functions and its derivatives of a fuzzy number is given in fuzzy functions. We let F denote the family of fuzzy numbers. A real number is determined by a fuzzy number

$$S_a = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases}$$

We can identify a fuzzy number $\tilde{\mu}$ in which the parameterized triples.

$$\{(a\,(r), b\,(r), r) | 0 \leq r \leq 1\} \qquad (1.1.1)$$

Where a(r) denotes the left hand endpoint of $c_r(\tilde{\mu})$ and b(r) denotes the right hand end point. and

$$c_r(\mu) = \begin{cases} \{x | \tilde{\mu}\,(x) \geq r\} & \text{if } 0 < r \leq 1, \\ \text{cl}\,(\text{supp}\,\hat{\mu}) & \text{if } r=0, \end{cases}$$

Where cl (supp $\hat{\mu}$) denotes the closure of the support of $\tilde{\mu}$

If $\tilde{\mu}$:R→I is a fuzzy number with parameterization give by (1.1.1). Then the functions a and b satisfy five conditions in [1]. Moreover, suppose that a:I →R and b:I→R satisfies the five conditions in (1.1.1). Then $\tilde{\mu}$:R→I defined by $\tilde{\mu}$(x)= sup{r| a (r) ≤ x ≤ b (r)} --------(1.1.2)

is a fuzzy number with parameterzation given by (1.1.1)

Let γ={{(a(r), b (r), r) | r∈I} a:I→R, b:I→R are bounded functions.

We define the addition, the scalar product and the metric on γ by (1.1.3), (1.1.4) and (1.1.5)

$$\{(a(r), b(r),r) | r\in I\} + \{(c(r),d(r),r) | r\in I\} = \{(a(r)+c(r),b(r)+d(r),r | r\in I\} \quad \ldots\ldots\ldots\ldots\ldots\ldots.\ (1.1.3)$$

$$c\{a(r), b(r),r) | r\in I\} = \{c(ca(r),cb(r),r) | r\in I\} \ldots\ldots\ldots\ldots\ldots\ldots.\ (1.1.4)$$

$$D(\{(a(r), b(r),r) | r\in I\}, \{(c(r), d(r),r) | r\in I\}).$$

$$= \sup\{\max\{|a(r)-c(r)|, |b(r)-d(r)\} r\in I\} \quad \ldots\ldots\ldots\ldots\ldots\ldots.\ (1.1.5)$$

It is clear that the vector space γ together with the metric form a topological vector space.

A function $\tilde{f}$:R→F is said to be a fuzzy function limit and continuity of fuzzy functions are studied with respect to the metric D defined by (1.1.5).

Theorem: Fuzzy function $\tilde{f}(X)$ = {(a (r,x), b(r,x), r) | r∈I } is continuous at x_0 ↔ The families {a(r,x) | r∈I } and {b(r,x) | r∈I }are equicontinuous with respect to x at x_0

Proof:

$\bar{f}(x)$ is continuous at x_0 ↔ for any given ∈ > 0 There exists a δ > 0 such that if $|x-x_0| < \delta$, then $D(\tilde{f}(X) = \tilde{f}(X_0)) < \in \leftrightarrow \forall \in > 0, \exists\, \delta > 0$, such that if $|x-x_0| < \delta$, then

$$|a(r,x) - a(r,x)| < \in$$
$$|b(r,x) - b(r,x)| < \in \text{ for any } r\in I$$

↔ families {a(r, x) | r∈I} and {b(r, x) | r∈I} are equicontinuous with respect to x at x_0.

Definition: Suppose that $\tilde{f}$:R→F and let x_0∈R. The derivative $\tilde{f}^1(X_0)$ of $\tilde{f}$ at the point x_0 is defined by

$$\tilde{f}^1(X_0) = \lim_{h\to 0} \frac{f(X_0 + h) - f(X_0)}{h}$$

There are two results for fuzzy derivative

- Suppose that $\tilde{f}$:R→F, x_0∈R and that $\lim_{h\to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$ exists

Let η = {(α(r), β(r), r| r∈I} be the parametric representation of this limit and for each x∈R, let{(a(r, x), b(r, x), r)| r∈I}be the parametric representation of $\tilde{f}$ (x). Then

  - ○ η∈γ and
  - ○ $\alpha(r) = a_x(r, x_0)$ and $\beta(r) = b_x(r, x_0)$ for each r∈I

  Where ax and b_x are the partial derivatives of a and b with respect to X.

- Suppose that $\tilde{f}$:R→F is a fuzzy function and that for each x, $\tilde{f}(X)$ is represented parametrically by {(a(r, x), b(r, x), r) | r∈I}. If a_x and b_x are continuous. Then $\tilde{f}'(x)$ exists for each x∈R

  The proof of following Theorem is straight forward.

Theorem: Suppose that $\tilde{f}(X)$ = {(a(r, x), b(r, x), r) | r∈I} is n^{th} differentiable fuzzy function, g(x) is n^{th} differentiable a real function, Then

$$[\tilde{f}.g]^{(n)} = \sum_{k=0}^{n} c_n^k \tilde{f}^{(n-k)} g^{(k)}$$

This result is similar to leibniz formula for distribution. Thus in this paper we only discuss fuzzy differential equations with real functions coefficients.

Fuzzy Differential Equations:

First we consider 1st order linear fuzzy differential equations.

$$\tilde{f}'(X) + p(X).\tilde{f}(X) = \tilde{g}(X) \qquad \text{------------------ (1.2.1)}$$

Where f(x) is a known fuzzy function with continuous derivative, g(x) is known continuous fuzzy function p(x) is a known real function.

Suppose that

$$\tilde{f}(x) = \{(\alpha(r, x), \beta(r, x), r) \mid r \in I\} \qquad \text{------------------ (1.2.2)}$$

$$\tilde{g}(x) = \{(a(r, x), b(r, x), r) \mid r \in I\}$$

By (1) and (I) we have that

$$\alpha_x(r, x) + p(x)\,\alpha(r, x) = a(r, x) \qquad \text{-------------------(1.2.3)}$$

$$\beta_x(r, x) + p(x)\,\beta(r, x) = b(r, x) \qquad \text{-------------------(1.2.4)}$$

Their solutions are

$$\alpha(r, x) = e^{-\int_{x_0}^{x} p(t)dt}\left(\int_{x_0}^{x} a(r,t) e^{\int_{x_0}^{t} p(\xi)d\xi} dt + c_1(r)\right) \qquad \text{------------------ (1.2.5)}$$

$$\beta(r,x) = e^{-\int_{x_0}^{x} p(t)dt}\left(\int_{x_0}^{x} b(r,t) e^{\int_{x_0}^{t} p(\xi)d\xi} dt + c_2(r)\right) \qquad \text{------------------ (1.2.6)}$$

Where x_0 is constant, $c_1(r)$, $c_2(r)$ are integral constants. We can prove that $\tilde{f}(X)$ ={(α(r, x), (r, x), r) | r∈I} is a fuzzy function.

Where α(r, x) and β(r, x) are given by (1.2.5) and (1.2.6) respectively {$c_1(r)$, $c_2(r)$, r) | r∈I} is a fuzzy number, $\tilde{f}(x)$ is general solution of the equation (1.2.1).

The initial value problem of fuzzy differential equation

$$(A)\quad \begin{cases} f'(x) + p(x) f(x) = g(x) \\ \tilde{f}(x)|_{x=x0} = \tilde{\mu} = \{(d(r), e(r), r) \mid r \in I\} \end{cases}$$

Can be changed into the initial-value problem of two ordinary differential equations.

$$(A\text{-}1)\quad \begin{cases} \alpha_x(r,x) + p(x)\alpha(r,x) = a(r,x) \\ \alpha_x(r,x_0) = d(r) \end{cases}$$

$$(A\text{-}2)\quad \begin{cases} \beta_x(r,x) + p(x)\alpha(r,x) = b(r,x) \\ \beta_x(r,x_0) = e(r) \end{cases}$$

The solution of (A-1) is

$$\alpha(r,x) = e^{-\int_{x_0}^{x} p(t)dt} \int_{x_0}^{x} a(r,t) e^{\int_{x_0}^{t} p(\xi)d\xi} dt + d(r))$$

And solution of (A-2) is

$$\beta(r,x) = e^{-\int_{x_0}^{x} p(t)dt}\left(\int_{x_0}^{x} b(r,t) e^{\int_{x_0}^{t} p(\xi)d\xi dt} dt + e(r)\right)$$

Now we put above two expressions into (1.2.2). The solution of (A) is obtained.

Example: Find the solution of the following initial-value problem of fuzzy differential equation.

$$\begin{cases} \tilde{f}'(x) + \dfrac{1}{50+x} f(x) = S_{24} \\ \tilde{f}(0) = S_{25} \end{cases}$$

Since S_{24} ={(24, 24, r) | r∈I}, S_{25}={(25, 25, r)|r∈I}

$$\int p(x)dx = \int \frac{dx}{50+x} = \text{In}(50+x)$$

$$\int a(r,x)\, e^{\int p\alpha d\alpha}\, dx = \int 24\,(50+x)dx = 1200\,x + 12x^2$$

$$\text{Thus } \alpha\,(r,x) = \frac{c_1(r) + 1200X + 12X^2}{50+X}$$

$$\beta\,(r,x) = \frac{c_2(r) + 1200X + 12X^2}{50+X}$$

Since $\alpha\,(r,0) = 25$ Thus $c_1(r)=1250$, similarly $c_2(r)=1250$. The solution of (A) is

$$\tilde{f}(x) = \left\{\left(\frac{12x^2+1200x+1250}{50+x}, \frac{12x^2+1200x+1250}{50+x}, r\right) \middle| r \in I \right\}$$

$$= s\left\{\frac{12x^2+1250x+1250}{50+x}\right\}$$

The example shows that ordinary differential equations are particular case of fuzzy differential Equations.

Example: Find the solution of the fuzzy differential equation.

Where $\tilde{f}^1(x) = \tilde{v}$

$$\tilde{v} = \begin{cases} 1-(t-a)^2, & \text{if } t \in [a-1, a+1] \\ 0, & \text{if } t \notin [a-1, a+1] \end{cases}$$

Where a is a real number.

It is obvious that $\tilde{v} = \left\{(a - \sqrt{1-r},\ a + \sqrt{1-r},\ r)\ |\ r \in I\right\}$

Let $\tilde{f}(x) = \left\{(\alpha(r,x), \beta(r,x), r)\ |\ r \in I\right\}$

We obtain two ordinary differential equations

$$\alpha_x(r,x) = a - \sqrt{1-r},\ \beta_x(r,x) = a + \sqrt{1-r}$$

Thus $\alpha(r,x) = (a - \sqrt{1-r})x + C_1(r),\ \beta(r,x) = (a + \sqrt{1-r})x + C_2(r)$

Hence $\tilde{f}(x) = \left\{((a - \sqrt{1-r})x + C_1(r),\ (a + \sqrt{1-r})x + C_2(r),\ r)\ |\ r \in I\right\}$

$= x\tilde{v} + \tilde{\mu}$

Where $\mu = \left\{(C_1(r),\ C_2(r), r)\ |\ r \in I\right\}$. The μ is arbitrary fuzzy constant.

The solution of the initial value problem

$$\begin{cases} \tilde{f}^1(x) = \tilde{v} \\ \tilde{f}(0) = 0 \end{cases} \text{ is , } \tilde{f}(x) = x\tilde{v}$$

Readers can easily see that the solution represents a motion of the particle with fuzzy number speed $\tilde{V}$.

To sum up, we can conclude that finding solutions a fuzzy differential equations with real function coefficient are actually solving two ordinary differential equations. Thus until nth order linear fuzzy differential equations with constant coefficient can be theoretically solved.

Solving Fuzzy Differential Equations - By Runge – Kutta Method:

Definition: Let x be a nonempty set. A fuzzy set u in x is characterized by its membership function u:x→ [0,1]. Then u(x) is interpreted as the degree of membership of a element x in the fuzzy set u for each $x \in X$. Let us denote by R_F the class of fuzzy subsets of the real axes (i.e. $u:R \to [0,1]$)

Satisfying the following properties

- $\forall u \in R_F$, u is normal. ie.$\exists x_0 \in R$ with $u(x_0)=1$
- $\forall u \in R_F$, u is convex fuzzy set
  (i.e. $u(tx+(1-t)y) \geq \min\{u(x), u(y)\}, \forall t \in [0,1]\}, x, y \in R$)

- $\forall u \in R_F$, u is upper semicontinuous on R.
- $cl\{x \in R;\ u(x) > 0\}$ is compact, where cl(A) denotes the closure of subsets A.

Definition:

Let x, $y \in R_F$. If there exists $z \in R_F$ such that x = y+z then z is called the H-difference of x, y and it is denoted by $x \Theta y$. Note that $x\Theta y \neq x+(-1)y = x-y$. In what follows we fix I = (a, b) for a, $b \in R$. Bede in [6] introduced a more general definition of a derivative for a fuzzy number valued function. In this paper we consider the following definition.

Definition: Let $f: I \to R_F$ be given. Fix $t_0 \in I$. We say f is (1) -differentiable at t_0 and its derivative denoted by $D_1 f$, If there exists an elements $f'(t_o) \varepsilon R_F$ such that for all h > 0 sufficiently small, there exist $f(t_0+h)\Theta f(t_0)$, $f(t_0)\Theta f(t_0-h)$ and the following limits.

$$\lim_{h\to 0^+} \frac{f(t_0+h)\Theta f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\Theta f(t_0-h)}{h} = f'(t_0)$$

similarly a function f is (2) -differentiable at t_0 and its derivative denoted by $D_2 f$, if there exists an element $f'(t_0) \in R_F$ such that for all h > 0 sufficiently small, there exist $f(t_0+h)\Theta f(t_0)$, $f(t_0)\Theta f(t_0-h)$ and the following limits.

$$\lim_{h\to 0^+} \frac{f(t_0+h)\Theta f(t_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0)\Theta f(t_0-h)}{h} = f'(t_0)$$

Generalized Characterization Theorem:

Let us consider the fuzzy differential equations with initial value condition.

$$x'(t) = f(t,x),\ x(t_0) = x_0 \qquad (2.1)$$

Where $f:[t_0, T]\times R_F \to R_F$ is a continuous fuzzy mapping and $x_0 \in R_F$ and T is positive number or infinity.

Definition: Let $y: I \to R_F$ be a fuzzy function such that $D_1 y$ or $D_1 y$ exists. If y and $D_1 y$ satisfy problem (2.1), we say y is a (2.1) - solution of problem (2.1). Similarly, if y and $D_2 y$. Satisfy Problem (2.1), we say y is a (2) solution of problem (2.1).

Let us suppose α-cut of functions x(t), x_0, f(t, x) are the following form.

$$[x(t)]^\alpha = [\underline{x}_\alpha(t),\ \overline{x}_\alpha(t)],$$

$$[x_0]^\alpha = [\underline{x}_0,\ \overline{x}_0]$$

$$[f(t, x(t))]^\alpha = [f_\alpha(t, x_\alpha,\ \overline{x}_\alpha),\ \overline{f}_\alpha(t, \underline{x}_\alpha, \overline{x}_\alpha)]$$

Case (1): If x(t) is (1) – differentiable then solving FIVP (2.1) translates into the following algorithm.

Step (i): Solving the following system of ODES.

$$\begin{cases} \underline{x}'_\alpha(t) = \overline{f}_\alpha(t, \underline{x}_\alpha, \overline{x_\varepsilon}) = G(t, \underline{x}_\alpha, \overline{x}), \underline{x}(t_0) = \underline{x}_0 \\ \overline{x}'_\varepsilon(t) = f_\alpha(t, \underline{x}_\alpha, \overline{x_\varepsilon}) = F(t, \underline{x}, \overline{x}), \underline{x}(t_0) = \overline{x}_0 \end{cases} \qquad (2.2)$$

Step (ii): Ensure that the solution [$\underline{x}_\alpha$ (t), $\overline{x}_\alpha$ (t)] and [$\underline{x}'_\alpha$ (t), $\overline{x}'_\alpha$ (t)] are valid level sets.

Step (iii): By using the representation theorem again,

We construct a(2.1)- solution x(t) such that $[x(t)]^\alpha$= [$\underline{x}_\alpha$ (t), $\overline{x}_\alpha$ (t)], for all $\alpha \in [0,1]$.

Case (ii): If x(t) is (2.2) - differentiable then.

Step (i): Solving FIVP (1) translates into the following algorithm step (i) solving the following system of ODES,

$$\begin{cases} \underline{x}'_\alpha(t) = \overline{f}_\alpha(t, \underline{x}_\alpha, \overline{x_\alpha}) = G(t, \underline{x}_\alpha, \overline{x}), \underline{x}(t_0) = \underline{x}_0 \\ \overline{x}'_\alpha(t) = f_\alpha(t, \underline{x}_\alpha, \overline{x_\alpha}) = F(t, \underline{x}, \overline{x}), \underline{x}(t_0) = \overline{x}_0 \end{cases} \qquad (2.3)$$

Step (ii): Ensure that the solution [$\underline{x}_\alpha$ (t), $\overline{x}_\alpha$ (t)] and [$\overline{x}'_\alpha$ (t), $\overline{x}'_\alpha$ (t)] are valid level sets.

Step (iii): By using the representation theorem again, we construct a(2.2)- solution x(t)such that $[x(t)]^\alpha$ = [$\underline{x}_\alpha$ (t), $\overline{x}_\alpha$ (t)], for all $\alpha \in [0,1]$.

Now we extend Bede's characterization to fuzzy differential equation under generalized differentiability.

Runge-Kutta Method for FDE:

Theorem: Under appropriate conditions, the FIVP (2.1) considered under generalized differentiability has locally two solutions and the successive iterations.

$$x(0) = x_0, \qquad x_{n+1}(t) = x_0 + \int_{t_0}^{T} f(s, x_n(s))\, ds$$

and

$$x(0) = x_0, \qquad x_{n+1}(t) = x_0 - (-1)\int_{t_0}^{T} f(s, x_n(s))\, ds$$

Converge to the (2.1) – solution and the (2.2) – solution respectively.

Proof: The authors of proved for (2.1)-differentiability. The result for (2.2)-differentiability is obtained the generalized characterization. Theorem, we replace the fuzzy differential equation with its equivalent system and then, for approximating the two fuzzy solutions. We solve numerically two ODE systems which consist of four classic ordinary differential equations with initial conditions. Now we extend Runge-Kutta method finding two fuzzy solutions of FDES under generalized differentiability. We consider the partition P for interval $[t_0, T]$.

$$p : t_0 = a_0 < a_1 < \ldots, < a_N = T$$

$$a_i = a_0 + ih,\ h = \frac{T - t_0}{N}$$

Suppose two exact solutions $[Y_1(t)]^{\infty} = \left[\underline{Y_1}(t, \infty),\ \overline{Y_1}(t, \infty)\right]$ and $[Y_2(t)]^{\infty} = \left[\underline{Y_2}(t, \infty),\ \overline{Y_2}(t, \infty)\right]$ are approximated by some $[Y_1(t)]^{\infty} = \left[\underline{Y_1}(t, \infty),\ \overline{Y_1}(t, \infty)\right]$, $[Y_2(t)]^{\infty} = \left[\underline{Y_2}(t, \infty),\ \overline{Y_2}(t, \infty)\right]$ respectively.

The exact and approximate solution at grid point a_i, $0 \le i \le N$ are denoted by $Y_{1n}(\alpha), Y_{2n}(\alpha), Y_{1n}(\infty)$ and $Y_{2n}(\infty)$ respectively.

The generalized Runge-Kutta method based on the second order approximation of $\underline{Y_1'}(t, \infty),\ \overline{Y_1'}(t, \infty)$, $\underline{Y_2'}(t, \infty),\ \overline{Y_2'}(t, \infty)$ and equations (2.2) and (2.3) is obtained as follows.

$$\begin{cases} \underline{Y_1}_{n+1}(\infty) = \underline{Y_1}_n(\infty) + \left(1 - \dfrac{1}{2\theta}\right) hF(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)) \\ \qquad\qquad + \left(\dfrac{1}{2\theta}\right) hF\left(t_n + \theta h, \underline{Z_1}^{\infty}_{nH}(\infty), \overline{Z}^{\infty}_{1n+1}\right) \\ \overline{Y}_{1n+1}(\infty) = \overline{Y_{1n}}(\infty) + \left(1 - \dfrac{1}{2\theta}\right) hG(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)) \\ \qquad\qquad + \left(\dfrac{1}{2\theta}\right) hG\left(t_n + \theta h, \underline{Z_1}^{\infty}_{nH}(\infty), \overline{Z}^{\infty}_{1n+1}\right) \\ \underline{Y_{10}}(\infty) = \underline{Y_0}(\infty) \\ \overline{Y}_{10}(\infty) = \overline{Y_0}(\infty) \end{cases} \quad (2.4)$$

$$\begin{cases} \underline{Z_1}^{\infty}_{n+1} = \underline{Y_1}_n(\infty) + \theta hF(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)) \\ \overline{\overline{Z}^{\infty}_{1n+1}} = \overline{Y_{1n}}(\infty) + \theta hG(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)) \end{cases} \quad (2.5)$$

$$\begin{cases} \underline{Y_2}_{n+1}(\infty) = \underline{Y_2}_n(\infty) + \left(1 - \dfrac{1}{2\theta}\right) hG(t_n, \underline{Y_2}_n(\infty), \overline{Y}_{2n}(\infty)) \\ \qquad\qquad + \left(\dfrac{1}{2\theta}\right) hG\left(t_n + \theta h, \underline{Z_2}^{\infty}_{n+1}(\infty), \overline{Z}^{\infty}_{2n+1}\right) \\ \overline{Y}_{2n+1}(\infty) = \overline{Y_{2n}}(\infty) + \left(1 - \dfrac{1}{2\theta}\right) hF(t_n, \underline{Y_2}_n(\infty), \overline{Y}_{2n}(\infty)) \\ \qquad\qquad + \left(\dfrac{1}{2\theta}\right) hF\left(t_n + \theta h, \underline{Z_2}^{\infty}_{n+1}(\infty), \overline{Z}^{\infty}_{2n+1}\right) \\ \underline{Y_2}_0(\infty) = \underline{Y_0}(\infty) \\ \overline{Y}_{20}(\infty) = \overline{Y_0}(\infty) \end{cases} \quad (2.6)$$

$$\begin{cases} \underline{Z_2}^{\infty}_{n+1} = \underline{Y_2}_n(\infty) + \theta hG(tn, \underline{Y_2}_n(\infty), \overline{Y}_{2n}(\infty)) \\ \overline{Z}^{\infty}_{2n+1} = \overline{Y_{2n}}(\infty) + \theta hF(tn, \underline{Y_2}_n(\infty), \overline{Y}_{2n}(\infty)) \end{cases} \qquad (2.7)$$

Lemma: Let the sequences of numbers $\{W_n\}_{n=0}^{\alpha}$, $\{V_n\}_{n=0}^{\alpha}$ satisfy

$|w_{n+1}| \leq |w_n| + A \max \{|w_n|+|V_n|\}+B$
$|V_{n+1}| \leq |V_n| + A \max \{|w_n|+|V_n|\}+B$
for some positive constants A and B and denote

$U_n = |w_n| + |V_n|, 0 \leq n \leq N$ then,

$$U_n \leq (1+2A)^n U_0 + 2B \frac{(1+2A)^n - 1}{(1+2A)-1}, \ 1 \leq n \leq N$$

The following theorem shows that the generalized Runge Kutta approximation pointwise converge to the exact solutions.

Let F(t,u,v) and G(t,u,v) be the functions F and G of equations (2.2) and (2.3) where u and v are constants and $u \leq V$. The domain where F and G are defined is therefore.

$k=\{(t,uv,|0\leq t\leq A,-\alpha<v<\alpha, -\alpha<u\alpha v\}$

Theorem: Let F(t, u, v) and G(t, u, v) belong C′(K) and let the partial derivatives of F and G be bounded over K. Then for arbitrary fixed $\alpha : \alpha \in [0, 1]$. The generalized Runge – Kutta approximation of equations (2.4) & (2.6) converge to the exact solution $Y_1(t, \infty)$, $Y_2(t, \infty)$ uniformly in it.

Proof: If we consider (2.1) differentiability, then for convergence of Equation (2.4) similar to (2.7) is sufficient to show.

$$\lim_{h \to 0} \underline{Y_1}_N(\infty) = Y_1(t, \infty), \ \lim_{h \to 0} \overline{Y}_{1N}(\infty) = \overline{Y}_1(t, \infty)$$

By using the Taylor Theorem we have

$$\underline{Y_1}_{n+1}(\infty) = \underline{Y_1}_n(\infty) + \left(1 - \frac{1}{20}\right) hF(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty))$$
$$+ \left(\frac{1}{20}\right) hF\left(t_n + \theta h, \underline{Z_1}^{\infty}_{n+1}, \overline{Z}^{\infty}_{1n+1}\right) + \frac{h^3}{6} \underline{Y_1}'''\left(\underline{\xi}_{1n}\right)$$

and

$$\overline{Y}_{1n+1}(\infty) = \overline{Y}_{1n}(\infty) + \left(1 - \frac{1}{2\theta}\right) hG(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty))$$
$$+ \left(\frac{1}{2\theta}\right) hG\left(t_n + \theta h, \underline{Z_1}^{\infty}_{n+1}, \overline{Z}^{\infty}_{1n+1}\right) + \frac{h^3}{6} \overline{Y}_1'''\left(\overline{\xi}_{1n}\right)$$

Where $t_n \leq \underline{\xi}_{1n}, \overline{\xi}_{1n} \leq t_{n+1}$ then we have

$$\underline{Y_1}_{n+1}(\infty) - \underline{Y_1}_{n+1}(\infty) = \underline{Y_1}_n(\infty) - \underline{Y_1}_n(\infty) + \left(1 - \frac{1}{2\theta}\right) h\left\{F\left[(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)\right]\right.$$
$$-F\left[(t_n, \overline{Y}_{1n}(\infty), \overline{Y}_{1n}(\infty)\right] + \left(\frac{1}{2\theta}\right) h\left\{F\left(t_n + \theta h, \underline{Z_1}^{\infty}_{n+1}, \overline{Z}^{\infty}_{1n+1}\right)\right.$$
$$\left.- F\left(t_n + \theta h, \underline{Z_1}^{\infty}_{n+1}, \overline{Z}^{\infty}_{1n+1}\right)\right\} + \frac{h^3}{6} \overline{Y}_1'''\left(\overline{\xi}_{1n}\right)$$

$$\overline{Y}_{1n+1}(\infty) - \overline{Y}_{1n+1}(\infty) = \overline{Y}_{1n}(\infty) - \overline{Y}_{1n}(\infty) + \left(1 - \frac{1}{2\theta}\right) h\left\{G\left[(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)\right]\right.$$
$$-G\left[(t_n, \underline{Y_1}_n(\infty), \overline{Y}_{1n}(\infty)\right] + \left(\frac{1}{2\theta}\right) h\left\{G\left(t_n + \theta h, \underline{Z_1}^{\infty}_{n+1}, \overline{Z}^{\infty}_{1n+1}\right)\right.$$

$$- G\left(t_n + \theta h,\ \underline{Z}_{1_{n+1}}^{\infty},\ \overline{Z}_{1n+1}^{\infty}\right)\Big\} + \frac{h^3}{6}\overline{Y}_1'''\left(\overline{\xi}_{1n}\right)$$

Similarly we have

$$\underline{Z}_{1_{n+1}}(\infty) - \underline{Z}_{1_{n+1}}(\infty) = \underline{Y}_{1_n}(\infty) - \underline{Y}_{1_n}(\infty) + \theta h\Big\{F\Big[(t_n,\ \underline{Y}_{1_n}(\infty),\ \overline{Y}_{1n}(\infty)\Big]$$

$$-F\Big[(t_n,\ \overline{Y}_{1n}(\infty),\ \overline{Y}_{1n}(\infty)\Big]\Big\} + \frac{h^2}{2}Y_1''\left(\underline{\eta}_{1_n}\right)$$

and

$$\overline{Z}_{1n+1}(\infty) - \overline{Z}_{1n+1}(\infty) = \overline{Y}_{1n}(\infty) - \overline{Y}_{1n}(\infty) + \theta h\Big\{G\Big[(t_n,\ \underline{Y}_{1_n}(\infty),\ \overline{Y}_{1n}(\infty)\Big]$$

$$-G\Big[(t_n,\ \underline{Y}_{1_n}(\infty),\ \underline{Y}_{1_n}(\infty)\Big]\Big\} + \frac{h^2}{2}\overline{Y}_1''\left(\overline{\eta}_{1n}\right)$$

Where $t_n \leq \underline{\eta}_{1_n}$, $\overline{\eta}_{1n} = t_{n+1}$

Now we define W_{1n}, V_{1n}, P_{1n}, T_{1n} by the following terms.

$$W_{1n} = \underline{Y}_{1_{n+1}}(\infty) - \underline{Y}_{1_{n+1}}(\infty), \quad V_{1n} = \overline{Y}_{1n+1}(\infty) - \overline{Y}_{1n+1}(\infty),$$

$$P_{1n} = \underline{Z}_{1_{n+1}}(\infty) - \underline{Z}_{1_{n+1}}(\infty), \quad T_{1n} = \overline{Z}_{1n+1}(\infty) - \overline{Z}_{1n+1}(\infty)$$

Then we have

$$\left|W_{1_{n+1}}\right| \leq \left|W_{1_n}\right| + \left(1 - \frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\}$$

$$+ \left(\frac{1}{2\theta}\right)2Lh\max\left\{\left|P_{1_{n+1}}\right|,\ \left|T_{1_{n+1}}\right|\right\} + \frac{h^3}{6}\underline{N}_1$$

$$\left|V_{1_{n+1}}\right| \leq \left|V_{1_n}\right| + \left(1 - \frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\}$$

$$+ \left(\frac{1}{2\theta}\right)2Lh\max\left\{\left|P_{1_{n+1}}\right|,\ \left|T_{1_{n+1}}\right|\right\} + \frac{h^3}{6}\overline{N}_1$$

$$\left|P_{1_{n+1}}\right| \leq \left|W_{1_n}\right| + 2Lh\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\} + \frac{h^2}{2}\underline{M}_1$$

$$\left|T_{1_{n+1}}\right| \leq \left|V_{1_n}\right| + 2Lh\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\} + \frac{h^2}{2}\overline{M}_1$$

where $\underline{N}_1 = \text{Sup}\ \overline{Y'''}_1(t, \infty)$, $\overline{N}_1 = \text{Sup}\ \overline{Y'''}_1(t, \infty)$, $\underline{M}_1 = \text{Sup}\ \underline{Y''}_1(t, \infty)$, $\overline{M}_1 = \text{Sup}\ \overline{Y''}_1\ (t, \infty)$ and L > 0 is a bound for the partial derivatives of F, G.

By substitute $\left|P_{1_{n+1}}\right|$, $\left|T_{1_{n+1}}\right|$ in $\left|W_{1_{n+1}}\right|$, $\left|V_{1_{n+1}}\right|$, we have

$$\left|W_{1_{n+1}}\right| \leq \left|W_{1_n}\right| + \left(1 - \frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|, \left|V_{1_n}\right|\right\} + \left(\frac{1}{2\theta}\right)2Lh\max\Big\{\max\left\{\left|W_{1_n}\right|, \left|V_{1_n}\right|\right\}$$

$$+ 2h\theta\left\{\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\} + \frac{h^2}{2}K_1\right\}\Big\} + \frac{h^3}{6}\underline{N}_1$$

and

$$\left|V_{1_{n+1}}\right| \leq \left|V_{1_n}\right| + \left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\} + \left(\frac{1}{2\theta}\right)2Lh\max\left\{\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right.$$

$$\left. + 2h\theta\left\{\max\left\{\left|W_{1_n}\right|,\ \left|V_{1_n}\right|\right\} + \frac{h^2}{2}K_1\right\}\right\} + \frac{h^3}{6}\overline{N_1}$$

where $K_1 = \max\left\{\underline{M_1},\ \overline{M_1}\right\}$. Now the above term can abbreviate to the following

$$\left|W_{1_{n+1}}\right| \leq \left|W_{1_n}\right| + \frac{h^3}{6}\left(N_1 + \frac{3L}{\theta}K_1\right) + \left(\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}\right)\left\{\left(1-\frac{1}{2\theta}\right)2Lh + \left(\frac{1}{2\theta}\right)2Lh(1+2Lh)\right\}$$

and

$$\left|V_{1_{n+1}}\right| \leq \left|V_{1_n}\right| + \frac{h^3}{6}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\left(1-\frac{1}{2\theta}\right)2Lh\max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\} + \max\left\{\left|W_{1_n}\right|,\left|V_{1_n}\right|\right\}$$

$$\left\{\left(1-\frac{1}{2\theta}\right)2Lh + \left(\frac{1}{2\theta}\right)2Lh(1+2Lh)\right\}$$

Then by Lemma 2.2.1 we have

$$\left|W_{1n}\right| \leq \left(1+4Lh(1+Lh)\right)^n\left|V_0\right| + \frac{h^3}{3}\left(\underline{N_1} + \frac{3L}{\theta}K_1\right)\frac{(1+4Lh(1+Lh)^n - 1}{4Lh(1+Lh)}$$

$$\left|V_{1n}\right| \leq \left(1+4Lh(1+Lh)\right)^n\left|V_0\right| + \frac{h^3}{3}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\frac{(1+4Lh(1+Lh)^n - 1}{4Lh(1+Lh)}$$

where $\left|V_0\right| = \left|W_{10}\right| + \left|V_{10}\right|$ in particular.

$$\left|W_{1n}\right| \leq \left(1+4Lh(1+Lh)\right)^N\left|V_0\right| + \frac{h^3}{3}\left(\underline{N_1} + \frac{3L}{\theta}K_1\right)\frac{(1+4Lh(1+Lh))^{(T-t_0)/h} - 1}{4Lh(1+Lh)}$$

$$\left|V_{1n}\right| \leq \left(1+4Lh(1+Lh)\right)^N\left|V_0\right| + \frac{h^3}{3}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\frac{(1+4Lh(1+Lh))^{(T-t_0)/h} - 1}{4Lh(1+Lh)}$$

Since $W_{10} = V_{10}$ and know for $\delta > -1$, relationship $e^{K\delta} > (1 + \delta)^K$ satisfy, then by assumption $K = \frac{T - t_0}{h}$, $\delta = 4Lh(1 + Lh)$ we have

$$\left|W_{1n}\right| \leq + \frac{h^3}{3}\left(\underline{N_1} + \frac{3L}{\theta}K_1\right)\frac{e^{4L(1+Lh)(T-t_0)} - 1}{4Lh(1+Lh)}$$

$$\left|V_{1n}\right| \leq \frac{h^3}{3}\left(\overline{N_1} + \frac{3L}{\theta}K_1\right)\frac{e^{4L(1+Lh)(T-t_0)} - 1}{4Lh(1+Lh)}$$

And if $h \to 0$ we get $W_N \to 0$, $V_N \to 0$ which concludes the proof.

Another Approach to Solution of Fuzzy Differential Equations:

A trapezoidal fuzzy number u is defined by four real numbers k < *l* < m < n, where the base of the trapezoidal is the interval [k, n] and its vertices at x = *l*, x = m, trapezoidal fuzzy number will be written as u = (k, *l*, m, n). The membership function for the trapezoidal fuzzy number u = (K, l, m, n) is defined as the following.

$$u(x) = \begin{cases} \dfrac{x - k}{l - k}, & K \leq x \leq l \\ 1, & l \leq x \leq m \\ \dfrac{x - n}{m - n}, & m \leq x \leq n \end{cases} \qquad \ldots (3.1)$$

We will have

- $u > 0$ if $K > 0$;
- $u > 0$ if $l > 0$;
- $u > 0$ if $m > 0$; and
- $u > 0$ if $n > 0$;

Let us denote R_F by the class of all fuzzy subsets of R(ie. $u : R \to [0, 1]$)
Satisfying the following properties:

- $\forall\ u \in R_F$, u is normal, i.e. for all $x_0 \in R$ with $u(x_0) = 1$.
- $\forall\ u \in R_F$, u is convex fuzzy set, i.e. $u(tx + (1-t)y \geq \min\{u(x), u(y)\}$, $\forall\ t \in [0, 1]$, $x, y \in R)$;
- $\forall\ u \in R_F$, u is upper semi continuous on R ;
- $\{x \in R ; u(x) > 0\}$ is compact, where $\overline{A}$ denotes the closure of A.

Then R_F is called the space of fuzzy numbers obviously $R \subset R_F$. Here $R \subset R_F$ is under stood as
$R = \{\{x\} ;$ x is usual real number$\}$
We define the r-level set, $x \in K$;
$[u]_r = \{x|u(x) \geq r\}\ 0 \leq r \leq 1$; . . . (3.2)

Clearly, $[u]_0 = \{x|u(x) > 0\}$ is compact

Which is closed bounded interval and we denote by $[u]_r = \left[\underline{u}(r), \overline{u}(r)\right]$. It is clear that the following statements are true.

- $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0, 1].
- $\overline{u}(r)$ is a bounded right continuous non increasing function over [0, 1].
- $\underline{u}(r) \leq \overline{u}(r)$ for all $r \in [0, 1]$, for more details see [1], [2].

Let $D : R_F \times R_F \to R_+ u(0)$,
$D(u, v) = \text{Sup}_{r \in [0, 1]} \text{Max}\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ be Laudorff distance between fuzzy numbers, where $[u]_r = [\underline{u}(r), \overline{u}(r)]$, $[v]_r = [\underline{v}(r), \overline{v}(r)]$.
The following properties are well known
$D(u+w, v+w) = D(u, v)$, $\forall\ u, v, w \in R_F$.
$D(k.u, k.v) = |K|D(u, v)$, $\forall\ K \in R$, $u,v \in R_F$.
$D(u+v, w+e) \leq D(u, w) + D(v, e)$, $\forall\ u, v, w, e \in R_F$ and (R_F, D) is a complete metric space.

Lemma: If the sequence of non-negative numbers $\{W_n\}_{n=0}^{N}$ satisfy $|w_{n+1}| \leq A|w_n| + B$, $0 \leq n \leq N - 1$, for the given positive constants A and B, then $|w_n| \leq A^n|w_0| + B\dfrac{A^n - 1}{A - 1}$, $0 \leq n \leq N$.

Lemma: If the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy $|W_{n+1}| \leq |W_n| + A\max\{|W_n|, |V_n|\} + B$,
$|V_{n+1}| \leq |V_n| + A\max\{|W_n|, |V_n|\} + B$. for the given positive constants A and B, then denoting
$U_n = |W_n| + |V_n|$, $0 \leq n \leq N$

We have, $U_n \leq \overline{A}^n u_0 + \overline{B}\dfrac{\overline{A}^n - 1}{\overline{A} - 1}$, $0 \leq n \leq N$,

Where $\overline{A} = 1+2A$ and $\overline{B} = 2B$

Lemma: Let F(t, u, v) and G(t, u, v) belong to $C^1(R_F)$ and the partial derivatives of F and G be bounded over R_F. Then for arbitrarily fired r, $0 \leq r \leq 1$

$$D(y(t_{n+1}), y^{(0)}(t_{n+1})) \leq h^2 L(1+2C)$$

Where L is a bound of partial derivatives of F and G and $C = \max\{G[|t_N, y(t_N:r), \overline{y}\ (t_{N-1}:r)]|, r \in [0, 1]] < \infty$

Fuzzy Initial Value Problem:

Consider a first order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), t\varepsilon[t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (3.1.1)$$

Where y is a fuzzy function of t, f(t, y) is a fuzzy function of the crisp variable t and the fuzzy variable y, y^1 is the fuzzy derivative of y and $y(t_0) = y_0$ is a trapezoidal or a trapezoidal shaped fuzzy number.

We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for $t\varepsilon[t_0, T]$ is

$[y(t)]r = [\underline{y}(t:r), \overline{y}(t:r)]$,

$[y(t_0)]r = [\underline{y}(t_0:r), \overline{y}(t_0:r)]$, $r\varepsilon(0,1]$

We write $f(t,y)=[\underline{f}(t, y), \bar{f}(t, y)]$ and

$$\underline{f}(t,y) = F[t,y,\bar{y}],$$

$$\bar{f}(t,y) = G[t,y,\bar{y}]$$

Because of $y' = f(t,y)$ we have

$$\underline{f}(t,y(t).r) = F[t,y(t:r), \bar{y}(t:r)] \qquad (3.1.2)$$

$$\bar{f}(t,y(t).r) = G[t,y(t:r), \bar{y}(t:r)] \qquad (3.1.3)$$

By using the extension principle, we have the membership function.

$$f(t, y(t)(s) = \sup\{y(t)(\iota) \mid s = f(t, \iota)\},\ s \in R \qquad (3.1.4)$$

So fuzzy number f(t, y(t). From this it follow that

$$[f(t, y(t)]_r = [f(t, y(t); r), \bar{f}(t, y(t):r), r\varepsilon(0,1] \qquad (3.1.5)$$

Where

$$\underline{f}(t, y(t);r) = \min\{f(t, y) \mid u \in [y(t)]_r\} \qquad (3.1.6)$$

$$\bar{f}(t, y(t); r) = \max\{f(t, u) \mid u \in [y(t)]_r\} \qquad (3.1.7)$$

Definition: A function $f:R \to R_F$ is said to be fuzzy continuous function, if for an arbitrary fixed $t_0 \in R$ and $\in > 0$, $\delta > 0$ such that $|t-t_0| < \delta \Rightarrow D[f(t),f(t_0) < \in$ exists throughout this chapter we also consider fuzzy functions which are continuous in metric D. Then the continuity of f(t, y(t); r) guarantees the extent of the definition of f(t, y(t); r) for $t\varepsilon[t_o, T]$ and $r \in [0, 1]$ [4]. Therefore, the functions G and F can be definite too.

Runge-Kutta Method of Order Three:

Consider the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \qquad (3.2.1)$$

Assuming the following Runge Kutta Method with three slopes

$$y(tn+1) = y(tn) + W_1K_1 + W_2K_2 + W_3K_3 \qquad (3.2.2)$$

Where $K_1 = hf(t_n, y(t_n))$

$$K_2 = hf(t_n + C_2h, y(t_n) + a_{21}K_1)$$

$$K_3 = hf(t_n + C_3h, y(t_n) + a_{31}K_1 + a_{32}K_2)$$

and the parameters $W_1, W_2, W_3, C_1, C_2, C_3, a_{21}, a_{31}$ & a_{32} are chosen to make y_{n+1} closer to $y(t_{n+1})$. There are eight parameters to be determined. Now, Taylors' series expansion about t_n, gives

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + \frac{h}{1!}y'(t_n) + \frac{h^2}{2!}y''(t_n) + \frac{h^3}{3!}y'''(t_n) + \ldots \\ &= y(t_n) + \frac{h}{1!}f(t_n, y(t_n)) + \frac{h^2}{2!}[f_1 + ff_y]_{t_n} \\ &\qquad + \frac{h^3}{3!}[f_{tt} + 2ff_{ty} + f^2f_{yy} + f_y(f_t + ff_y)]_{t_n} + \ldots \end{aligned} \qquad (3.2.3)$$

If we set

$K_1 = hf_n$

$K_2 = hf(t_n + C_2h, y(t_n) + a_{21}K_1)$

$$K_2 = h\left\{(f_n + \frac{h}{1!}\left[C_2f_t + a_{21}ff_y\right]_{t_n} + \frac{h^2}{2!}\left[C_2^2f_{tt} + 2C_2a_{21}ff_{ty} + a_{21}^2f^2ff_{ty}\right]_{t_n} + \ldots\right\}$$

$K_3 = hf(t_n + C_3h, y(t_n) + a_{31}K_1 + a_{32}K_2)$

$$\begin{aligned} &= \Big\{f_n + C_3hf_t + \left[a_{31}K_1 + a_{32}K_2\right]f_y + \frac{1}{2!}\Big[C_3^2h^2f_{tt} + 2C_3h(a_{31}K_1 + a_{32}K_2)f_{ty} \\ &\qquad + (a_{31}K_1 + a_{32}K_2)^2f_{yy} + \ldots\Big]\Big\} \end{aligned}$$

$$\begin{aligned} &= h\Big\{f_n + \frac{h}{1!}\left[C_3f_t + (a_{31} + a_{32})f_nf_y\right]_{t_n} + \frac{h^2}{2!}\Big[2(C_2f_t + a_{21}ff_y)a_{32}f_y) \\ &+ C_3^2f_{tt} + 2C_3a_{31}f_n + 2C_3a_{32}f_{ty}f_n + \left(a_{31}^2 + a_{32}^2 + 2a_{31}a_{32}\right)f_n^2f_{yy}\Big]_{t_n} \end{aligned}$$

$$+\frac{h^3}{3!}\Big[3(C_2^2f_{tt}+2C_2a_{21}ff_{ty}+a_{21}^2f^2f_{yy})a_{32}f_y+(6C_3a_{32}f_{ty}+6a_{31}f_na_{32}f_{yy})$$

$$(C_2f_t+a_{21}f_{yy})\Big]_{t_n}+\ldots\Big\}$$

Substituting the values of K_1, K_2 & K_3 in (3.2.2) we get

$$y(t_{n+1})=y(t_n)+\left[W_1+W_2+W_3\right]hf_n+h^2\left[W_2\left(C_2f_t+a_{21}ff_y\right)+W_3\left(C_3f_t\left(a_{31}+a_{32}\right)f_nf_y\right)f_nf_y\right]_{t_n}$$

$$+\frac{h^3}{2}\Big[W_2\left(C_2^2f_{tt}+2C_2a_{21}ff_{ty}+a_{21}^2f^2f_{yy}\right)+W_3\Big(2\left(C_2f_t+a_{21}ff_{yy}\right)a_{32}f_y$$

$$+C_3^2f_{tt}+2C_3a_{31}f_n+2C_3a_{32}f_nf_{ty}+\left(a_{31}^2+a_{32}^2+2a_{31}a_{32}\right)f_n^2f_{yy}+\ldots\Big)\Big]_{t_n}\quad(3.2.4)$$

Comparing the coefficients of h, h^2 & h^3 in (3.2.3) & (3.2.4), we obtain

$$a_{21}=C_2,\qquad a_{31}+a_{32}=C_3,\qquad W_1+W_2+W_3=1$$

$$C_2W_2+C_3W_3=\frac{1}{2}\qquad C_3^2W_2+C_2^2W_3=\frac{1}{3}\qquad C_3a_{32}W_3=\frac{1}{6}\qquad(3.2.5)$$

Multiplying the fourth and fifth equations by C_2Q_{32} and using the sixth equation of (3.2.5) we get

$$C_3^2a_{32}W_2+C_3\left(\frac{1}{6}\right)=\frac{1}{2}C_2a_{32},\quad C_2^3a_{32}W_2+C_3^2\left(\frac{1}{6}\right)=\frac{1}{3}C_2a_{32}$$

Eliminating W_2 from these two equations, we find that no solution exists unless

$$\frac{3C_2a_{32}-C_3}{6C_2^2a_{32}}=\frac{2C_2a_{32}-C_3^2}{6C_2^3a_{32}}\quad(\text{or})\quad a_{32}=\frac{C_3\left(C_3-C_2\right)}{C_2\left(2-3C_2\right)}\qquad(3.2.6)$$

Usually, C_2, C_3 are arbitrarily chosen and a_{32} is determined from (3.2.6). However, if $C_2 = C_3$, then we immediately obtain from the fourth and fifth equations of (3.2.5), that $C_2 = \frac{2}{3}$. The values of the remaining parameters are obtained from (3.2.5)

When $C_2 = C_3$, we get $C_2 = \frac{2}{3}$ and $a_{21} = \frac{2}{3}$. We get the values of the other parameters as $a_{31} = 0$, $a_{32} = \frac{2}{3}$, $W_1 = \frac{2}{8}$, $W_2 = \frac{3}{8}$ and $W_3 = \frac{3}{8}$.

Runge-Kutta Method is obtained as

$$y(t_{n+1})=y(t_n)+\frac{1}{8}\left[2K_1+3K_2+3K_3\right]\qquad(3.2.7)$$

Where $K_1 = hf(t_n, y(t_n))$

$$K_2=hf\left(t_n+\frac{2h}{3},\ y(t_n)+\frac{2}{3}K_1\right)$$

$$K_3=hf\left(t_n+\frac{2h}{3},\ y(t_n)+\frac{2}{3}K_2\right)\ldots$$

Runge-Kutta Method of Order Three for Solving Fuzzy Differential Equations:

Let $Y=\left[\underline{Y},\ \overline{Y}\right]$ be the exact solution and $Y=\left[\underline{Y},\ \overline{Y}\right]$ be the approximated solution of the fuzzy initial value problem.

Let $[Y(t)]_r=\left[\underline{Y}(t\,.\,r),\ \overline{Y}(t\,.\,r)\right],\ \left[Y(t)\right]_r=\left[\underline{Y}(t\,.\,r),\ \overline{Y}(t\,.\,r)\right]$

Throughout this argument the value of r is fixed. Then, the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)] = \left[\underline{Y}(t_n \,.\, r), \overline{Y}(t_n \,.\, r)\right]$$

$$[Y(t_n)]_r = \left[\underline{Y}(t_n \,.\, r), \overline{Y}(t_n \,.\, r)\right] \quad (0 \leq n \leq N).$$

The grid points at which the solution is calculated are $h = \dfrac{T - t_0}{N}$,

$t_i = t_0 + ih,\ 0 \leq i \leq N$
Then we obtain,

$$\underline{Y}(t_{n+1}\,;r) = \underline{Y}(t_{n+1}\,;r) + \frac{1}{8}\left[2K_1 + 3K_2 + 3K_3\right] \text{ where}$$

$$K_1 = hF\left[t_n, \underline{Y}(t_n\,;r), \overline{Y}(t_n\,;r)\right]$$

$$K_2 = hF\left[t_n + \frac{2h}{3}, \underline{Y}(t_n\,;r) + \frac{2}{3}K_1, \overline{Y}(t_n\,;r) + \frac{2}{3}K_1\right] \tag{3.3.1}$$

$$K_3 = hF\left[t_n + \frac{2h}{3}, \underline{Y}(t_n, r) + \frac{2}{3}K_2, \overline{Y}(t_n\,;r) + \frac{2}{3}K_2\right]$$

And $$\overline{Y}(t_{n+1}\,;r) = \overline{Y}(t_n\,;r) + \frac{1}{8}\left[2K_1 + 3K_2 + 3K_3\right]$$

Where

$$K_1 = hG\left[t_n, \underline{Y}(t_n\,;r), \overline{Y}(t_n\,;r)\right]$$

$$K_2 = hG\left[t_n + \frac{2h}{3}, \underline{Y}(t_n\,;r) + \frac{2}{3}K_1, \overline{Y}(t_n\,;r) + \frac{2}{3}K_1\right] \tag{3.3.2}$$

$$K_3 = hG\left[t_n + \frac{2h}{3}, \underline{Y}(t_n, r) + \frac{2}{3}K_2, \overline{Y}(t_n\,;r) + \frac{2}{3}K_2\right]$$

Also we have,

$$\underline{Y}(t_{n+1}\,;r) = \underline{Y}(t_n\,;r) + \frac{1}{8}\left[2K_1 + 3K_2 + 3K_3\right]$$

Where

$$K_1 = hF\left[t_n, \underline{Y}(t_n\,;r), \overline{Y}(t_n\,;r)\right]$$

$$K_2 = hF\left[t_n + \frac{2h}{3}, \underline{Y}(t_n\,;r) + \frac{2}{3}K_1, \overline{Y}(t_n\,;r) + \frac{2}{3}K_1\right] \tag{3.3.3}$$

$$K_3 = hF\left[t_n + \frac{2h}{3}, \underline{Y}(t_n, r) + \frac{2}{3}K_2, \overline{Y}(t_n\,;r) + \frac{2}{3}K_2\right]$$

And $$\overline{Y}(t_{n+1}\,;r) = \overline{Y}(t_n\,;r) + \frac{1}{8}\left[2K_1 + 3K_2 + 3K_3\right]$$

Where

$$K_1 = hG\left[t_n, \underline{Y}(t_n\,;r), \overline{Y}(t_n\,;r)\right]$$

$$K_2 = hG\left[t_n + \frac{2h}{3}, \underline{Y}(t_n\,;r) + \frac{2}{3}K_1, \overline{Y}(t_n\,;r) + \frac{2}{3}K_1\right] \tag{3.3.4}$$

$$K_3 = hG\left[t_n + \frac{2h}{3}, \underline{Y}(t_n, r) + \frac{2}{3}K_2, \overline{Y}(t_n \; ; r) + \frac{2}{3}K_2 \right]$$

Clearly, $\underline{Y}(t \; ; r)$ and $\overline{Y}(t \; ; r)$ converge to $\underline{Y}(t \; ; r)$ and $\overline{Y}(t \; ; r)$ respectively whenever $h \to 0$.

Numerical Results:

In this section, the exact solutions and approximated solutions obtained by Euler's method and Runge Kutta method of order three are plotted in tables.

Example: Consider the initial value problem

$$\begin{cases} y'(t) = f(t)), t\varepsilon[0,1] \\ y(0) = (0.8 + 0.125r, 1.1 - 0.1r) \end{cases}$$

The exact solution at t=1 is given by y(1:r)] = [(0.8+0.125r)e, (1.1– 0.1r)e], 0 ≤ r ≤ 1

Using iterative solution of Range-Kutta method of order three, we have

$$\underline{y}(0;r) = 0.8+0.125r,$$

$$\underline{y}(0;r) = 1.1 - 0.1r$$

and by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}(t_i;r) + h\underline{y}(t_i;r),$$

$$\overline{y}^{(0)}(t_{i+1};r) = \overline{y}(t_i;r) + h\overline{y}(t_i;r),$$

Where i= 0,1,.........N–1 and h=1/N. Now , using equations as an initial guess for following interative solutions respectively,

$$y^j(t_{i+1}; r) = y(t_i; r) + \frac{1}{8}[2k_1+3k_2+3k_3],$$

Where

$$k_1 = h\,\underline{y}\,(t_i; r)$$

$$k_2 = h\,\underline{y}\,(t_i; r) + \frac{2}{3}k_1)$$

$$k_3 = h\,\underline{y}\,(t_i; r) + \frac{2}{3}k_2)$$

and $\overline{y}^j(t_{i+1};r) = \overline{y}\;(t_i;r) + \frac{1}{8}[2k_1 + 3k_2 + 3k_3]$

Where $k_1 = h\,\overline{y}\,(t_i; r)$

$$k_2 = h\,\overline{y}(t_i;r) + \frac{2}{3}k_1)$$

$$k_3 = h\,\overline{y}(t_i;r) + \frac{2}{3}k_2\,)$$

And j = 1,2,3 Thus we have $\underline{y}(t_i : r) = y^{(3)} = (t_i : r)$

And $\overline{y}(t_i : r) = \overline{y}^{(3)}(t_i : r),$ for i = 1.........N

Therefore , $\underline{y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\overline{y}(1 : r) \approx \underline{y}^{(3)}(1;r)$ are obtained.

Table : 3 Shows estimation of error for different values of r∈[0, 1] and h. By minimizing the step size h, the solution by exact method and RK method almost coincides

r	**Exact solution**
0	2.174625, 2.990110
0.2	2.242583, 2.935744
0.4	2.310540, 2.881379
0.6	2.378497, 2.827013
0.8	2.446454, 2.772647
1	2.514411, 2.718282

Table 1: Exact solution

r h	0.1	0.01
0	1.958468, 2.692893	2.174515, 2.989958
0.2	2.019670, 2.643931	2.242468, 2.935595
0.4	2.080872, 2.594970	2.310422, 2.881232
0.6	2.142074, 2.546008	2.378875, 2.826869
0.8	2.203276, 2.497046	2.446329, 2.772506
1	2.264478, 2.448085	2.514283, 2.718143

Table 2: Approximated solution

r h	0.1	0.01
0	0.513374	0.000262
0.2	0.514726	0.000264
0.4	0.516077	0.000265
0.6	0.517428	0.000266
0.8	0.518779	0.000266
1	0.520130	0.000267

Table 3: Error for different values of r and h

A New Fuzzy Version of Euler's Method for Solving Differential Equations with Fuzzy Initial Values:

The Extension Principle:

The idea of the extension principle is easy to understand. Let f be a function that maps from X to Y. The extension principle provides a mechanism to transform a fuzzy set defined in X to a fuzzy set defined in Y.

Let F(X) and F(Y) be the sets of all fuzzy sets defined in X and Y respectively and f : X → Y be a continuous function. The function f induces a mapping f : F(X) → F(Y) such that if A is a fuzzy sets in X, then it ranges under f is a fuzzy set B = f(A) whose membership function is expressed as in the following equation

$$f(A)(Y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x), & \text{if } y \in \text{range}(f), \\ 0, & \text{if } y \in \text{range}(f), \end{cases}$$

Where

$$f^{-1}(y) = \{x \in X \mid f(x) = y\} \quad \text{(inverse of f)}$$

Roman – Flores et al (2001) have show that if f : x → y is a continuous function, then f : F(x) → F(y) is a well-defined function, and

$$[F(A)]^{\infty} = F([A]^{\infty}),$$

for all $\infty \in [0, 1]$ and $A \in f(x)$.

Fuzzy Initial Value Problems:

In this section, we first consider the following ordinary differential equation.

$$\begin{cases} x'(t) = f(t, x(t)), t\varepsilon[t_0, T] \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

Where $f:[t_0,T]xR \to R$ is a continuous function defined on $[t_0,T]$ with $t > 0$ and $x_0 \in R$. Suppose that the initial condition in (4.1) is uncertain and modeled by a fuzzy interval, then we have the following fuzzy initial value problem.

$$\begin{cases} x'(t) = f(t, x(t)), t\varepsilon[t_0, T] \\ x(t_0) = x_0 \end{cases} \tag{4.2}$$

Where $f:[t_0,T]\times F(R) \to F(R)$ is fuzzy valued function defined on $[t_0,T]$ with $T > 0$ and $x_0F(R)$. To interpret the connection between (4.1) and (4.2), we refer to mizukoshi 2007 and hullermerier (1997).

Interpretation under Zadeh's Extension Principle:

Let U be an open set in R such that there exists a solution $x(.,x_0)$ of (10) with $x_0 \in U$ on the interval $[t_0,T]$ and for all $t \in [t_0,T]$, $x(t,.)$ is continuous in U. Then we can define

$$x(t, x_0) : U \to R$$

Which is the unique fuzzy solution of (10). If x_0 replace by x_0, which is a fuzzy interval, then from Zadeh's extension principle we have

$$x(t, x_0) : F(U) \to F®$$

Which is the unique fuzzy solution of (11).

Interpretation under Hullermeier's Approach:

In agreement with Hullermeier (1997), the differential equation in (4.2) can be interpreted as follow

$$\begin{cases} x'_\beta(t) = f(t, x_\beta(t)), t\varepsilon[t_0, T] \\ x_\beta(t_0) = [x_0]^\beta \end{cases} \qquad (4.3)$$

Where f: $[t_0,T]xR \rightarrow F(R)$ is a real – valued function defined on $[t_0,T]$ with T > 0 and $\beta\varepsilon[t_0, 1]$

For every $\beta\varepsilon[0,1]$ we say that $x_\beta : [t_0, T] \rightarrow R$ is the β - solution of (12) if it is absolutely continuous and satisfies (4.3) almost everywhere on $[t_0T]$ with T > 0, let M_β be the set of all β -solution of (4.3) and we define the attainable sets as

$$A_\beta(t) = \{x_\beta(t) | x_\beta(.) \varepsilon M_\beta\}$$

Which is the β - cut of fuzzy attainable set A(t). Hence, the fuzzy attainable set is the solution of (4.2) Chalco-Cano and Roman – Flores (2008) have proven that the solution obtained by Zadeh's extension principle concides with the solutions obtained by Hullermeier's approach.

A New Fuzzy Version of Euler's Method:

We recall Taylor's theorem in order to derive the classical Euler method. Suppose that x(t), the unique solution of (4.1) have two continuous derivatives on the interval $[t_0, T]$,
so that for each j = 0, 1, 2, . . . N-1,

$$x(t_{j+1}) = x(t_j) + (t_{j+1} - t_j)Y'(t_j) + \frac{(t_{j+1} - t_j)^2}{2} Y''(\xi_j) \qquad . . . (4.4)$$

For some numbers $\xi_j \in (t_j, t_{j+1})$. By setting $h = t_{j+1} - t_j$,
We have that

$$x(t_{j+1}) = x(t_j) + hx'(t_j) + \frac{h^2}{2} x''(\xi_j) \qquad . . . (4.5)$$

And since x(t) satisfies the problem (4.1), we have

$$x(t_{j+1}) = x(t_j) + hf(t_j, x(t_j)) + \frac{h^2}{2} x''(\xi_j) \qquad . . . (4.6)$$

By truncating the remainder term and denoting $x_j \approx x(t_j)$, then we have the following Euler Method for the problem (4.1)

$$x_{j+1} = x_j + hf(t_j, x_j) \qquad . . . (4.7)$$

for each j = 0, 1, 2 , . . . , N – 1.

In order to extend the classical Euler method (4.7) in fuzzy setting, we need to take into account the depending problem among fuzzy sets. First, let us consider the following situation

$$M(h, t, x) = x + hf(t, x) \qquad . . . (4.8)$$

If $x \in F(R)$. Then (4.8) can be extended in fuzzy setting as follow

$$M(h, t, x) = \begin{cases} \sup\limits_{x \in M^{-1}(h,t,z)} X(x), & \text{if } Z \in \text{range } (M) \\ 0, & \text{if } z \notin \text{range } (M) \end{cases} \qquad . . . (4.9)$$

Since (4.9) has very complicated structure, then we can solve it by using the ∞-cut of fuzzy interval X. Let $(X)^\infty = [X_1^\infty, X_2^\infty]$ be the ∞-cuts of X for all $\infty \in (0, 1]$ then (4.9) can be computed as follows.

$$M(h, t, [x]^\infty) = [\min\{M(h, t, x) \mid x \in [X_1^\infty, X_2^\infty]\}$$
$$\max\{M(h, t, x) \mid x \in [X_1^\infty, X_2^\infty]\}] \qquad . . . (4.10)$$

By using this idea, we calculate the Euler Method (4.9) as follows.

$$X^\infty_{j+1,1} = \min\{(x + Gh(f, x)) \mid x \in [X^\infty_{j,1}, X^\infty_{j,2}]\}$$
$$X^\infty_{j+1,2} = \max\{(x + Gh(t, x)) \mid x \in [X^\infty_{j,1}, X^\infty_{j,2}]\}$$

To solve the minimum and maximum problems, we adopt a computational method proposed by Ahmed et al 2010. The method is described in the next section.

The Computational Method:

Let X = (a, b, c) be a triangular fuzzy interval. The ∞-cut of X is denoted by $[X]^\infty = [X_1^\infty, X_2^\infty]$ for $\infty \in [0, 1]$. First, we discretise ∞ is the form $\infty_0 < \infty_1 < . . . < \infty_{n+1} < \infty_n$, where $\infty_0 = 0$ and $\alpha_n = 1$. The discretised ∞

are equally spaced. That is $\infty_1 = \infty_0 + i\Delta h$, for i = 0, 1, 2, . . . , n and $\Delta h = \frac{1}{n} > 0$. In this study, Δh is called the discretisation spacing after discretisation, we have a set of ∞ with (n + 1) elements.

$$\infty = \{\infty_0, \ldots, \infty_i, \ldots, \infty_n\} \quad \ldots (4.11)$$

This leads to a set I of (n+1) closed intervals.

$$I = \{[X]^{\infty 0}, \ldots, [X]^{\infty i}, \ldots, [X]^{\infty n}\} \quad \ldots (4.12)$$

For the different ∞-cuts of X the following property holds.

$$[X]^{\infty i+1} \subseteq [X]^{\infty i} \ \forall \ \alpha_i, \infty_{i+1} \in [0, 1] \text{ with } \alpha_i \leq \alpha_{i+1} \quad \ldots (4.13)$$

for i = 0, 2, . . . , n-1 from (4.13) it is clear that the ∞-cut of A at ∞_{i+1} is subsets of the ∞-cutsof A at ∞_i (see figure 2).

Since this property true for all $\infty \in [0, 1]$, The ∞-cut of x can be constructed as the Union of sub-intervals as shown in the following equations.

$$[X]^{\infty i} = [X_1^{\infty i}, X_1^{\infty i+1}] \cup [X_1^{\infty i+1}, X_2^{\infty i+1}] \cup [X_2^{\infty i+1}, X_2^{\infty i}] \ldots (4.14)$$

In order to find the numerical solution of (ii), we computer B = (m, h, t, x) at each level of ∞_i for i = 1, 2, 3, . . . , n according to the following equations.

$$b_1^{\infty_i} = \min\left[\min_{x \in [x_1^{\infty i}, x_1^{\infty i+1}]} m(t, h, x), \min_{x \in [x_1^{\infty i+1}, x_2^{\infty i+1}]} m(h, t, x), \min_{x \in [x_2^{\infty i+1}, x_2^{\infty i}]} m(h, t, x), \right] \ldots (4.15)$$

$$b_2^{\infty_i} = \max\left[\max_{x \in [x_1^{\infty i}, x_1^{\infty i+1}]} m(h, t, x), \max_{x \in [x_1^{\infty i+1}, x_2^{\infty i+1}]} m(h, t, x), \max_{x \in [x_2^{\infty i+1}, x_2^{\infty i}]} m(h, t, x), \right] \ldots (4.16)$$

here $b_1^{\infty_i}$ and $b_2^{\infty_i}$ are the lower and upper bounds of B respectively at ∞_i for i = 0, 1, . . . , n. In order to interpolate the points ($b_1^{\infty_i}$, ∞_i) and ($b_2^{\infty_i}$, ∞_i) for all i = 0, 1, . . . , n. We use linear splin interpolation. Finally, a fuzzy interval B is obtained. This process is repeated for all $t_i \in [t_0, T]$ for j = 0, 1, . . . , N-1.

Conclusion:

Variation of constants formula we provided solutions to fuzzy initial value problems for first order linear fuzzy differential equations. These solutions may have decreasing length of their support. The examples provided in this paper show us that we can have in this case asymptotic behaviour of the solutions similarly to the crisp case, and also, we may have reversible processes which was not the case under H-differentiability. The disadvantage of strongly generalized differentiability of a function with respect to H-differentiability and Hukuhara differentiability seems to be that a fuzzy differential equation has not a unique solution. The advantage is that the solution better reflects the behaviour of real-world system. We have studied the numerical solution of differential equations with fuzzy initial values. By taking into account the dependency problem in fuzzy computation, we proposed a new version of Euler method, which is a generalisation of the conventional one. In order to show the capability of the proposed method, we conducted several numerical examples including linear and linear differential equations with fuzzy initial values. Final results showed that the numerical method proposed in this paper produced better solutions compared to the numerical method proposed in the literature.

References:

1. R. Goetschel & W.Voxman, Elementary fuzzy caloulus, fuzzy sets and systems.
2. G. A. Anastassion and S.G. Gal, on a fuzzy trigonometric approximation theorem of weierstrass type, J. fuzzy math. 9(2001), No: 3, 701-708.
3. B. Bede, B. G. Gal, Almost periodic fuzzy-number-valued functions, fuzzy sets and systems, 147, 2004, 385-403.
4. J. J. Buckley and E. Eslami, Introduction to fuzzy logic and fuzzy sets, physica verbg, Heidelberg, Germany.2001.
5. Y. Chalco-cano, H. Roman-Flores, on New solutions of fuzzy differential Equations, Chaos, solitons and Fractals 38, 112-119,2008.
6. O. Kaleva, fuzzy differential equations, fuzzy sets ans sytems 24,301-317, 1987.
7. Laksmikantham, V. 2004. Set Differential equations versus fuzzy differential equations, Applied Mathematics and Computation 164:277-294.
8. Ahamed, M. Z. & Hasan, M. K. 2010. Incorporating optimization technique into Euler method for solving sifferential equations with fuzzy initial values. Proceedings of the 1st Regional Conference on Applied and Engineering Mathmatics: 2-3 June, Penang, Malaysia.
9. S. Abbasbandy, T. Allahvinloo, Numerical solutions of fuzzy differential equations by Taylor method, Journal of Computational methods in applied Mathematics 2,113-124,2002

10. M. L. Puri and D.A. Ralescu, Differentials of Fuzzy Functions, J. Math. Anal. Appl., 91(1983)321-325.
11. S. Seikkala, on the fuzzy initial value problem, Fuzzy Sets and Systems, 24, (1987), 319-330.
12. Shiji Song, Congxin Wu, Existence and uniqueness of solutions to the Cauchy problem of fuzzy differential equations, Fuzzy sets and systems, 110(2000), 55-67.
13. Zadeh, L.A. 1975a. The concept of Linguistic variables and its applications to approximate reasoning I, Information Sciences 8:199-249.
14. Zadeh, L.A. 1975b. The concept of Linguistic variables and its applications to approximate reasoning II, Information Sciences 8:301-357.
15. Zadeh, L.A. 1965. Fuzzy Sets. Information and control 8: 338-353.