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THE INEQUIVALENCE OF THE BARBER'S PARADOX AND RUSSELL'S PARADOX AND THE SOLUTION TO THE BARBER'S PARADOX

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Abstract:

The Barber's Paradox and Russell's Paradox have existed and been discussed for many years. Some argue that the two paradoxes are equivalent. This article proves that they are not equivalent. Some mathematicians believe that the solution to the barber's paradox is "there is no such barber." This article proves that this solution is wrong. Some other people want to solve the paradox by proposing to redefine the rule of the barber to exclude the barber from being considered and therefore to avoid contradictions. But this is not to solve the problem, but to avoid the problem. Through detailed analysis and making definition of related concepts, this article compares the barber with his other customers, and concludes that the solution to the barber's paradox is: there is such a barber, the barber should shave himself but he can shave himself only once." Based on this solution, this article also puts forward innovative suggestions in the field of mathematics.

Key Words: paradox; barber's paradox; Russell's paradox; group; group theory; logic; logical contradiction; mathematics; set; set theory; the third mathematical crisis; mathematical crises; element; assemblage; class; collection; self-sustainable; black hole; singularity; simulation of black hole; calculation of black hole; changeable set; correlation between sets; related; unrelated; correlation; action; function of set; function

Introduction:

The Barber's Paradox and Russell's Paradox are famous paradoxes in the field of logic. Russell's Paradox has been put forward and discussed for more than 100 years by Bertrand Russell (and independently, Ernst Zermelo). It is described as follows:

Form now the assemblage of all classes which are not members of themselves. This is a class: is it a member of itself or not? If it is (it belongs to itself), it is one of those classes that are not members of themselves, i.e., it is not a member of itself (it doesn't belongs to itself). If it is not (it doesn't belongs to itself), it is not one of those classes that are not members of themselves, i.e. it is a member of itself (it belongs to itself). Thus of the two hypotheses – that it is, and that it is not, a member of itself – each implies its contradictory. This is a contradiction.

The barber's paradox refers to a paradox when a barber promises to provide shaving service to all people who don't shave themselves and not to provide shaving service to people who shave themselves. Such a promise raises the question of whether he should shave himself or not? If he shaves himself, he can't shave himself. If he doesn't shave himself, he must shave himself. Therefore, whether he shaves himself or not, there will be problems. In this way, a logical contradiction is formed, hence, the barber's paradox.

Some people consider the barber's paradox to be a popular way to describe Russell's paradox and believe that the two paradoxes are not equivalent.

Some argue that the barber's paradox is equivalent to Russell's paradox. Because, if each person corresponds to a set, the elements of this set are defined as the people that the person shaves. Then, the barber claims that the elements in his corresponding set are all people in the city that do not belong to his own corresponding set, and all people in the city who do not belong to his own corresponding set belong to the corresponding set of the barber, then does the barber belong to his own corresponding set? Therefore, the barber's paradox and Russell's paradox are equivalent to each other. This argument seems to be valid but the author will points out later inside this article that actually there is a serious flaw in it.

Regarding the solution to the barber's paradox, the mathematician QUINE once proposed a solution, that is, "there is no such a barber." This solution seems to be able to solve the problem, because proving that a problem has no solution is also a way to solve the problem.

However, the author believes that Quine's solution is wrong. At least from the current wording, the barber's paradox can be analyzed and its solution can be found. This problem is solvable, that is, "Such a barber exists."

In addition, there are other theories that claim to have solved the "barber's paradox." For example, some theories claim that a way to solve the "barber's paradox" is to modify the barber's rules to exclude himself from the rules, i.e., he is not one of the people that needs to be considered when he makes the above-mentioned promise. But it is meaningless to propose a solution in this way. Because it is tantamount to modifying the proposition so that the problem itself does not exist, but it is not a solution to the problem. It just avoids the problem.

Others have proposed a solution, proposing that the gender of the barber can be defined as female, so that there is no problem of shaving herself. This proposition seems to be able solve the problem, but in fact, this solution also excludes the barber herself from being considered. It is essentially the same as the previous solution. Therefore, it is just verbal mischief and it is meaningless.

For the barber’s paradox, the arguments proposed by the author is based on the true meaning of the paradox, direct detailed discussions on the real logical contradictions of the paradox is provided, and then a logical and practical solution is given. Also in this research article, the author will analyze whether the barber's paradox and Russell's paradox are equivalent.

Analyses and Arguments:

For ease of discussion, this article first works on finding the solution for the barber’s paradox.

In order to provide a solution, first of all, we need to look at the specific wording of the barber's paradox. Because, to a large extent, whether the barber's paradox can be solved or not depends on how it is worded. Differences in language expression can cause the paradox to be resolvable or not resolvable. Because, for any proposition, if its wording is different, it’s meaning can be different, and then it can become a proposition that can be solved or a proposition that cannot be solved. For example, in extreme case, if the wording of the barber's paradox is expressed as "a barber who cannot shave any human being, but he also shave himself, and he is a human being", then, this proposition described in such wording can't be solved. Therefore, we need to solve the barber’s paradox according to the actual, accurate and meaningful wording of the paradox.

Therefore, let’s take a look at the common expressions of the Barber’s Paradox:

There is a barber in a certain city. His advertising slogan reads like this: "My shaving skills are very superb and are well-known throughout the city. I will shave all the people in this city who don’t shave themselves, and I will shave only the people in this city who don’t shave themselves. I sincerely welcome you!" The people who came to him to shave were many, and naturally they were the ones who didn't shave themselves. However, one day, the barber saw in the mirror that his own beard was growing. He instinctively grabbed the razor. But do you think he can shave himself according to his slogan? If he doesn't shave himself, he is a "person who doesn't shave himself" and he has to shave himself. But if he shaves himself, he is a “person who shaves himself", so he shouldn't shave himself.

In this form of expression, the barber's paradox can easily be resolved. Because, if a barber has never shaved himself, then he clearly is a "person who does not shave himself", and clearly is not a "person who shaves himself". There is no doubt about this.

Then, he should pick up the razor without hesitation and shave himself.

After he shaved himself, he is no longer a" person who doesn't shave himself", but becomes a "person who shaves himself". Then, he shouldn't shave himself any more. And, from then on, he is always a" person who shaves himself" because he had done the action of shaving himself, although only once. Actually the number of times he has done the action of shaving himself is not important. Even if he only shaves himself once in his life, he is still a" person who shaves himself". Because, when defining "person who shaves themselves", apparently, a person who shaves himself once a day is a" person who shaves himself", a person who shaves himself once a year is still a" person who shaves himself", and a person who shaves himself once one hundred years is also a" person who shaves himself" as long as the action of shaving has happened.

In other words, the barber should shave himself if he has never shaved himself. After he has shaved himself once, he never should shave himself again.

Therefore, the solution to the barber's paradox is that the barber "should shave himself and should only shave himself once."

But, some people may argue that the barber will encounter a logical contradiction in the process of shaving himself. Because he is shaving himself, he belongs to the group of "people who shave themselves", then he should not shave himself. In this way, a logical contradiction arises. In order to solve this contradiction, it is necessary to define the meaning of "shaving".

For example, "shaving" can be defined as "cutting the entire beard on a person's face with a razor so that the length of each hair of the beard exposed out of the skin surface is not more than one millimetre." If defined in this way, only after the barber has shaved every hair of the beard on his face, he will belong to the group of "people who shave himself." Before that, he is still not a "person who shaves himself." Therefore, when he is in the process of shaving himself, there was no logical contradiction. He just needs to start shaving and finish it. When he completely finishes shaving, he becomes a "person who shaves himself".

According to this logic, we can also define "shaving" as "cutting at least N hairs of the beard on a person's face with a razor, so that the length of each already cut hair of the beard exposed out of the skin surface is not more than M millimetre". If defined in this way, only after the barber has cut all N hairs of the beard on his face, he will belong to the group of "people who shave himself." Before that, he is not a "person who shaves himself." And once he finishes cutting his N hairs of beard, he just stops shaving his face, that's it. Therefore, in the process of cutting his hairs of beard, there is no logical contradiction; he just needs to complete the cutting. When he completes the defined "shaving" action, he becomes a "person who shaves himself". In extreme case, we take N as 1. Then, if the barber had never cut even one single hair of his beard, he should shave himself, i.e., he should cut 1 hair of his beard, then he should stop, and never cut his beard again.

Therefore, we can conclude, no matter how we define "shaving", the solution is always that the barber "should shave himself and should only shave himself once."

This solution is actually not very complicated or difficult to understand. But why many people didn't think of it?

It is probably because, when people think about this paradox, the concepts are confused. It needs to be pointed out that shaving is not an essential attribute of a person, but it is an action. Shaving is different from one's skin color, or one's gender, etc., which are considered to be the essential attributes of a person. Usually, essential attributes are considered unchanged for a person. If a person is born with yellow skin, he will have yellow skin all his life. If a person is born a male, then he will be a male all his life (when transgender practices are not considered). But shaving is different, it is just an action. So, if this action does not happen (a man hasn't shaved himself), he belongs to one category, i.e., he belongs to the group of people who don't shave themselves. If this action has already happened (he has shaved himself), he belongs to another category, i.e, he belongs to the group of people who shave themselves. Therefore, the barber can belong to different categories under different real situation.

The same logic applies not only to the barber himself, but also to his customers. For example, in this city, John is a customer of the barber, and he doesn't save himself and has been shaved by the barber for several years. But suddenly one day, John wants to try something different and pick up the razor and start to shave himself. Now what will happen? Will this cause a logical contradiction and make the barber in reality collapse into the void and disappear into the vast emptiness of the universe? Of course not. The barber only needs to stop providing shaving service for John because John is no longer a man who doesn't shave himself.

Then, the barber still doesn't violate his slogan and he is still a barber who "shaves all the people in this city who do not shave themselves and only shaves the people in this city who do not shave themselves". There is neither a logical contradiction nor a violation of the principles he sets for himself.

Therefore, as mentioned earlier, the solution to the barber's paradox can be determined as:

The barber "should shave himself and should only shave himself once."

As of now, the article starts to focus on whether the barber's paradox and Russell's paradox are equivalent.

As described earlier, barber's paradox can also be interpreted as:

If each person corresponds to a set, the elements of this set are defined as the people that the person shaves. Then, the barber claims that the elements in his corresponding set are all people in the city that do not belong to his own corresponding set, and all people in the city who do not belong to his own corresponding set belong to the corresponding set of the barber, then does the barber belong to his own corresponding set?

Described in this way, the barber's paradox seems identical to Russell's paradox. However, this is a serious flaw in this description of barber's paradox in real situation. That is, in reality, the meaning of "each person corresponds to a set, the elements of this set are defined as the people that the person shaves" is not specific. It's actually meaning is not clear so it is not well defined. Or we can just say directly that this description is wrong because it is not given a real meaning at all. It is just to put a bunch of words together and has not actual meaning. Actually there can be two realistically possible meaning of it. The first is "each person corresponds to a set; the elements of this set are defined as the people that the person has ever shaved". The second is "each person corresponds to a set; the elements of this set are defined as the people that the person promise to/plan to shave". Therefore, there are actually two different sets. The first is "each person corresponds to a set A; the elements of this set are defined as the people that the person has ever shaved". The second is "each person corresponds to a set B; the elements of this set are defined as the people that the person promise to/plan to shave".

Following these correct and logical definitions, the actual description of the paradox will be:

"Then, the barber claims that the elements in his corresponding set B are all people in the city that do not belong to his own corresponding set A, and all people in the city who do not belong to his own corresponding set A belong to the corresponding set B of the barber. "

Then if the barber has never shaved himself before, it is clear that he doesn't belong to his own corresponding set A, and he belongs to his own corresponding set B.

Because he belongs to his corresponding set B, he will shave himself as promised/planned. After the action of shaving himself has happened, he will belong to his own corresponding set A, and then he doesn't belong to his own corresponding set B.

This conclusion is also in line with the above-mentioned solution to the barber's paradox.

Now we can see, the barber's paradox is not equivalent to Russell's paradox because the set described in barber's paradox will be defined by different meanings, hence defined to be two different sets instead of one set that has a single reasonably defined meaning. Therefore both the content and form of the barber's paradox is no longer identical to the content and form of Russell's paradox. Thus apparently they are not equivalent.

At this point, some may argue, following the above-mentioned correct and logical definitions, there can be a different description for barber's paradox:

"Then, the barber claims that the elements in his corresponding set A are all people in the city that do not belong to his own corresponding set A, and all people in the city who do not belong to his own corresponding set A belong to the corresponding set A of the barber"

Following this line of reasoning, then if the barber has never shaved himself before, it is clear that he doesn't belong to his own corresponding set A, thus he belongs to his own corresponding set A, i.e., he has shaved himself. This is to say: if he has never shaved himself then he claims that he has shaved himself. This is not a logical contradiction, the truth is already clear that he has never shaved himself. The claim that he has shaved himself is false. It is pure meaningless lying.

Following the same reasoning, if the barber has shaved himself before, then he belong to his own corresponding set A, thus he doesn't belongs to the corresponding set A of the barber, i.e., he has not shaved himself. This is to say: if he has shaved himself then he claims that he has not shaved himself. Again this is not a logical contradiction. The fact that he has shaved himself is true and the claim that he has never shaved himself is false.

Thus apparently, the barber's paradox is not equivalent to Russell's paradox. Because the barber's paradox doesn't involve a logical contradiction, it just contains one true fact and one false claim, while Russell's paradox encounters a logical contradiction.

To sum up, the argument that the barber's paradox is equivalent to Russell's paradox is only a result of incorrectly description of barber's paradox. After we correctly define barber's paradox, it is clear that the barber's paradox is not equivalent to Russell's paradox

Conclusions:

- The solution to the barber's paradox can be determined as:
  The barber "should shave himself and should only shave himself once."
- The barber's paradox is not equivalent to Russell's paradox.

Suggestions:

Now that the conclusions have been reached, the author hopes that, based on the above discussions, constructive ideas can be suggested in the aspect of mathematical theories, so as to provide certain innovation opportunities for the development and evolution of current mathematical theories.

Specifically, the Barber's Paradox is related to Russell's Paradox and Russell's paradox questions the mathematical foundation of set theory, so the proposal of this solution may be related to potential innovation related with set theory. Also, because the essence of a paradox is a loss of sustainability, it is similar to some natural phenomena when sustainability is lost. For example, the process of forming a black hole singularity from a star due to immensely increasing gravity which far surpasses the supporting force of the star and causes the loss of sustainability. Hereby the author wishes to provide the following tentative suggestions to professionals just for reference.

- Is it possible to consider variable elements in set theory? In the existing set theory, the elements in a set are determined and unchangeable. When we discuss the barber's paradox, whether a person (including the barber himself) is "the one who shaves himself" or "the one who doesn't shave himself" is not always determined but changeable. At different times, when a certain action occurs, or when a certain incident happens, the object itself changes. According to the same logic, can we also consider variable elements in set theory? For example, the element itself can be a function $y=f(x)$. Different functions can be included as elements in a set and relevant mathematical theories can be established accordingly.
- Is it possible to consider uncertain elements in set theory? The previous suggestion focuses on the possibility of" the element which is determined but can be changeable under certain conditions". Here, the element itself can also be uncertain. For example, many physical properties of matter in quantum mechanics are uncertain. Moreover, some physical properties are simply not possible to be measured accurately; therefore they are not certain, under certain conditions. So, can a set theory system be established which allows uncertain elements?
- Is it possible to find some theoretical guidelines for paradox and use it for calculation and simulation of such natural phenomena as when a huge star collapse into a singularity of a black hole? Because paradox is not self-sustainable and the process of a star collapsing into a black hole singularity is also a process featuring the loss of self-sustainability.
- It should also be possible to establish an expression system for function of a set. For example, for set X, the function of set X can be expressed as $Y=f(X)$.
  When we quote our previous discussion inside this article:

"Each person corresponds to a set A; the elements of this set are defined as the people that the person has ever shaved".
And
"each person corresponds to a set B; the elements of this set are defined as the people that the person promise to/plan to shave"
Then for the barber, in the beginning when he has never shaved himself:
A doesn't include barber himself. B include barber himself.
When we define Y=f(X) as implementing the action that the barber shave all people inside the set, then, according to his promise to shave only the people who has never shaved themselves:
Y_A=f(A)=A
Y_B=f(B)=B-the barber, meaning to delete the barber from B.
After the action that the barber shave all people inside the set A and set B:
A'=A+the barber, meaning to add the barber as an element into set A, this is actually a result of f(B) instead of f(A)
B'=B-the barber, meaning to delete the barber from B.
Later, because the barber will only shave the people he promise to/plan to shave, then:
f(A')will never happen because he can not violate his promise,
$Y_{B'}$ =f(B')=B', unless any people inside the set take the action to shave himself.
To further explore, the computation equation of the following can be proposed when the elements inside f(X) correspond one by one to the elements inside set X:
f(M)∪f(N)=f((M∪N)
But apparently f(M)∩f(N)doesn't always equals f((M∩N)
Further computations are now open and can be discussed in the future.

- It could be possible or better to establish an expression system for implementation of a certain action under certain condition for a set or many sets. For example, for set X, the implementation of a certain action for it can be expressed as Y=action(X), or Y=a(X).
When we quote our previous discussion inside this article:
"Each person corresponds to a set A; the elements of this set are defined as the people that the person has ever shaved".
And
"each person corresponds to a set B; the elements of this set are defined as the people that the person promise to/plan to shave".
Then for the barber, when he has never shaved himself:
A doesn't include barber himself. B include barber himself.
When we define Y= action(X) as implementing the action that the barber shave all people inside the set, then, according to his promise to shave only the people who has never shaved themselves:
Y_A=action(A)=A
Y_B=action(B)=B-the barber, meaning to delete the barber from B.
After the action that the barber shave all people inside the set A and set B:
A'=A+the barber, meaning to add the barber as an element into set A, this is actually a result of, action(B) instead of action(A)
B'=B-the barber, meaning to delete the barber from B. Later, because the barber will only shave the people he promise to/plan to shave, then:
action (A') will never happen because he can not violate his promise,
$Y_{B'}$ =action (B')=B', unless any people inside the set take the action to shave himself.
To further explore, the computation equation of the following can be proposed when the elements inside Y=action(X) correspond one by one to the elements inside set X:
$Y_M \cup Y_N$=action(M)∪action(N)=action((M∪N)=$Y_{(M \cup N)}$
But apparently action (M)∩action(N) doesn't always equals action((M∩N)
Here, because action (B) can have an effect not only on B but also on A, so we can say:
set A and set B are RELATED.
Now, regarding the RELATIVITY between sets, we can conclude:
When considering set M and set N where as elements inside M and N are changeable, such as the set A and set B discussed inside this article, by a certain action according to certain conditions or principles,
if action(M) can have an effect on N, set M and set N are RELATED;
if action(N) can have an effect on M, set M and set N are RELATED;
if action(M) doesn't have an effect on N, and action(N) doesn't have an effect on M, then set M and set N are UNRELATED;
now
if set M and set N are RELATED, it can be express as:

M☰N
if set M and set N are UNRELATED, it can be express as:
M☷N
if set M and set N are RELATED, whereas action(M) can have an effect on N but action(N) can't have an effect on M, it can be express as:
M☰>N
Similarly, if set M and set N are RELATED, whereas action(N) can have an effect on M but action(M) can't have an effect on N, it can be express as:
M<☰N
Similarly, if set M and set N are RELATED, whereas action(N) can have an effect on M and action(M) can also have an effect on N, it can be express as:
M<☰>N
On this subject, the current discussions and suggestions are just a beginning, many points are still open and further research and discussions can be conducted.

Summaries:

- Quine's solution to the barber's paradox is wrong, because such a barber does exist.
- Revising the barber's rules to exclude himself from the rules can help to avoid the paradox, but did not solve the paradox.
- The solution to the barber's paradox is:
  The barber "should shave himself and should only shave himself once."
- The barber's paradox is not equivalent to Russell's paradox.
- It is suggested to consider variable elements and uncertain elements in set theory, and build relevant mathematical theories. It is also suggested to use the theories related with paradox for calculations and simulations of such natural phenomena as the process when a star collapse into the singularity of a black hole.
- It should be possible to establish an expression system for function of a set. For example, for set X, the function of set X can be expressed as Y=f(X).And it can be concluded that
  f(M)∪f(N)=f((M∪N)
  but apparently f(M)∩f(N) doesn't always equals f((M∩N)
- It could be possible or better to establish an expression system for implementation of a certain action for a set or many sets. For example, for set X, the implementation of a certain action for it can be expressed as Y=action(X), or a(X).And it can be concluded that
  $Y_M \cup Y_N$=action(M)∪action(N)=action((M∪N)=$Y_{(M \cup N)}$
  But apparently action (M)∩action(N) doesn't always equals action((M∩N)
- The CORRELATION between sets can be expressed as:
  When considering set M and set N whereas elements inside M and N are changeable by a certain action, according to certain conditions or principles:
  if action(M) can have an effect on N, set M and set N are RELATED;
  if action(N) can have an effect on M, set M and set N are RELATED;
  if action(M) doesn't have an effect on N, and action(N) doesn't have an effect on M, then set M and set N are UNRELATED;
  if set M and set N are RELATED, it can be express as:
  M☰N
  if set M and set N are UNRELATED, it can be express as:
  M☷N
  if set M and set N are RELATED, whereas action(M) can have an effect on N but action(N) can't have an effect on M, it can be express as:
  M☰>N
  if set M and set N are RELATED, whereas action(N) can have an effect on M but action(M) can't have an effect on N, it can be express as:
  M<☰N
  if set M and set N are RELATED, whereas action(N) can have an effect on M and action(M) can also have an effect on N, it can be express as:
  M<☰>N

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